On the perturbation of the group generalized inverse for a class of bounded operators in Banach spaces ☆

N. Castro-González *, J.Y. Vélez-Cerrada

Departamento de Matemática Aplicada, Facultad de Informática, Universidad Politécnica de Madrid, 28660 Boadilla del Monte, Madrid, Spain

Abstract

Given a bounded operator A on a Banach space X with Drazin inverse A^D and index r, we study the class of group invertible bounded operators B such that $I + A^D(B - A)$ is invertible and $\mathcal{R}(B) \cap \mathcal{N}(A^r) = \{0\}$. We show that they can be written with respect to the decomposition $X = \mathcal{R}(A^r) \oplus \mathcal{N}(A^r)$ as a matrix operator, $B = \begin{pmatrix} B_1 & B_{12} \\ B_{21} & B_{21}B_1^{-1}B_{12} \end{pmatrix}$, where B_1 and $B_1^2 + B_{12}B_{21}$ are invertible. Several characterizations of the perturbed operators are established, extending matrix results. We analyze the perturbation of the Drazin inverse and we provide explicit upper bounds of $\|B^\sharp - A^D\|$ and $\|BB^\sharp - A^D A\|$. We obtain a result on the continuity of the group inverse for operators on Banach spaces.

1. Introduction

Let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators on a complex Banach space X. For $A \in \mathcal{B}(X)$ we write $\mathcal{N}(A)$ for its kernel and $\mathcal{R}(A)$ for its range.

An operator $A \in \mathcal{B}(X)$ is *Drazin invertible* if there exists $C \in \mathcal{B}(X)$ such that

$$AC = CA, \qquad CAC = C, \qquad A^2C - A \text{ is nilpotent.} \tag{1.1}$$

In this case C is called a Drazin inverse of A. If A is Drazin invertible, then A has an unique Drazin inverse and is denoted by A^D. The *Drazin index* of A, ind (A), is equal to r if $A^2C - A$ is nilpotent of index r. If $\operatorname{ind}(A) = 1$, the A^D is denoted by $A^\sharp$ and is called the *group inverse* of A (see [2,14]).

It is well known that $A \in \mathcal{B}(X)$ is Drazin invertible with $\operatorname{ind}(A) = r \geqslant 1$ if and only if $\alpha(A) = \delta(A) = r$, where $\alpha(A)$ and $\delta(A)$ stand for the *ascent* and the *descent* of A, respectively. In this case the subspaces $\mathcal{R}(A^r)$ and $\mathcal{N}(A^r)$ are closed and X admits a topological decomposition $X = \mathcal{R}(A^r) \oplus \mathcal{N}(A^r)$.

We write $\sigma(A)$, $\rho(A)$ and $r(A)$ for the spectrum, the resolvent set and the spectral radius of A, respectively. For $\lambda \in \rho(A)$ we denote the resolvent $(\lambda I - A)^{-1}$ by $R(\lambda, A)$. If 0 is an isolated point of $\sigma(A)$, then the *spectral projection* of A associated with $\{0\}$ is defined by

$$A^{\pi} = \frac{1}{2\pi i} \int_{\gamma} R(\lambda, A)\, d\lambda,$$

where γ is a small circle surrounding 0 and separating 0 from $\sigma(A) \setminus \{0\}$.

If A is Drazin invertible, $\operatorname{ind}(A) = r$, the following well-known properties hold:

(i) $\mathcal{R}(A^D) = \mathcal{R}(A^r)$ and $\mathcal{N}(A^D) = \mathcal{N}(A^r)$.

(ii) $A^{\pi} = I - AA^D$, and so $\mathcal{N}(A^{\pi}) = \mathcal{R}(A^r)$ and $\mathcal{R}(A^{\pi}) = \mathcal{N}(A^r)$. In the particular case that $\operatorname{ind}(A) = 1$ then $AA^{\pi} = 0$.

In [2] is shown that $A \in \mathcal{B}(X)$ is Drazin invertible and $\operatorname{ind}(A) = r \geqslant 1$ if and only if $R(\lambda, A)$ has a pole of order r at $\lambda = 0$. In this case we have

$$R(\lambda, A) = \sum_{n=1}^{r} \frac{A^{n-1}A^{\pi}}{\lambda^n} - \sum_{n=0}^{\infty} \lambda^n \left(A^D\right)^{n+1} \tag{1.2}$$

in the region $0 < |\lambda| < (r(A^D))^{-1}$.

There are certain kinds of bounded operators T (“simply polaroid” [8]) for which every time $0 \in \sigma(T)$ is an isolated point it is a simple pole: for example the paranormals for which $\|Tx\|^2 \leqslant \|T^2x\|\|x\|$, in particular hermitian and normal operators on Banach spaces, and hyponormal operators on Hilbert spaces [18].

Operators with rational resolvents are Drazin invertible; necessary and sufficient to be group invertible is that 0 is at worst a simple pole of the resolvent.

We introduce the class of group invertible operators $B \in \mathcal{B}(X)$ such that

$$I + A^D(B - A) \text{ is invertible}, \qquad \mathcal{R}(B) \cap \mathcal{N}\left(A^r\right) = \{0\}. \tag{1.3}$$

In terms of the spectral projections associated with $\{0\}$, they can be described by the condition

$$I - B^{\pi} - A^{\pi} \text{ invertible}. \tag{1.4}$$

Under an equivalent assumption, the perturbation of the spectral projections in the setting of elements of Banach algebras was investigated in [6] with an approach different from the one given in this work. The particular case $B^{\pi} = A^{\pi}$ for closed operators was considered in [4]. A related paper is [7], which contains characterizations of bounded operators with equal projections related to their outer or inner generalized inverse.

The class of operators that we study induce the following space decompositions:

$$X = \mathcal{R}\left(A^r\right) \oplus \mathcal{N}(B), \qquad X = \mathcal{R}(B) \oplus \mathcal{N}\left(A^r\right).$$

If we assume that X is a Hilbert space of finite dimension, then these conditions turn out to be equivalent to

$$\mathcal{R}\left(A^r\right) \cap \mathcal{N}(B) = \{0\}, \qquad \mathcal{R}(B) \cap \mathcal{N}\left(A^r\right) = \{0\}.$$

The class of matrices which satisfy the above relations was studied in [5]. The purpose of this paper is to extend the perturbation analysis of Drazin inverses for matrices to bounded operators. Some of the results are new even in the case of matrices. The continuity and the perturbation of the Drazin inverse for linear operators and elements of Banach algebras were investigated in [3,10,15,16,20].

The paper is organized as follows. In Section 2 we deal with operators acting on the Banach space $X = X_1 \oplus X_2$ with matrix form as follows:

$$T = \begin{pmatrix} T_1 & T_{12} \\ T_{21} & T_{21}T_1^{-1}T_{12} \end{pmatrix}, \quad T_1 \text{ invertible in } X_1. \tag{1.5}$$

Our aim is to characterize the operator matrices of the above form which are group invertible and derive a formula for the group inverse of T. With this in mind we develop a representation for the resolvent of T.

In Section 3, we give several characterizations of the class of group invertible operators $B \in \mathcal{B}(X)$ which verify the conditions in (1.3). Apart from the facts we have mentioned before, we see that $(I + A^D(B - A))^{-1}A^D$ is an algebraic generalized inverse of B. We show that the perturbed operator B can be written with respect to the topological decomposition $X = \mathcal{R}(A^r) \oplus \mathcal{N}(A^r)$ as a matrix operator in the form (1.5), where the entries satisfy, in addition, that $B_1^2 + B_{12}B_{21}$ is invertible.

In Section 4 we analyze the perturbation of the Drazin inverse. The results developed in the previous sections, Theorems 2.2 and 3.4, are used to obtain upper bounds of $\|B^\sharp - A^D\|$ and $\|BB^\sharp - A^DA\|$. As an application of the previous results we give a necessary and sufficient condition for the continuity of the group inverse for bounded operators.

In [1,12,13] is showed that the group inverse plays an important role in the analysis of Markov chains. Recently, in [19] is showed that the group inverse provides a unifying framework for discrete-time Markov chains with kernels treated as bounded linear operators between spaces of measures. Our results could be applied to the perturbation theory of the Markov chains with kernels defined by bounded operators.

We finish this introduction giving some basic facts about algebraic generalized inverses which will be used in the paper (see [14]).

Let $T \in \mathcal{B}(X)$. An operator $T^g \in \mathcal{B}(X)$ is called an *inner inverse* of T if $TT^gT = T$ and an *outer inverse* if $T^gTT^g = T^g$. If T^g is both an inner and an outer inverse of T, the T^g is called an *algebraic generalized inverse* or *AG-inverse*.

Recall that $T \in \mathcal{B}(X)$ has an AG-inverse $T^g \in \mathcal{B}(X)$ if and only if $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are closed and have topological complements in X. In this case, we have:

(i) TT^g and T^gT are bounded projectors such that
$$\mathcal{R}(TT^g) = \mathcal{R}(T), \qquad \mathcal{R}(T^gT) = \mathcal{R}(T^g), \qquad \mathcal{N}(TT^g) = \mathcal{N}(T^g), \qquad \mathcal{N}(T^gT) = \mathcal{N}(T);$$
(ii) $X = \mathcal{N}(T) \oplus \mathcal{R}(T^g)$, $X = \mathcal{N}(T^g) \oplus \mathcal{R}(T)$.

2. The group inverse of a class of bounded operator matrices

In this section we deal with operator matrices $T = \begin{pmatrix} T_1 & T_{12} \\ T_{21} & T_2 \end{pmatrix}$ considered on the Banach space $X = X_1 \oplus X_2$. The operator T_i acts on the Banach space X_i, $i = 1, 2$, and the operator T_{12} (respectively T_{21}) acts from X_2 to X_1 (respectively from X_1 to X_2). We assume that all entries are bounded linear operators between the corresponding spaces.

Theorem 2.1. *Assume that $T \in \mathcal{B}(X)$ has the operator matrix form $T = \begin{pmatrix} T_1 & T_{12} \\ T_{21} & T_2 \end{pmatrix}$, where T_1 is invertible and $T_2 = T_{21}T_1^{-1}T_{12}$, and let $\Psi = I + T_1^{-1}T_{12}T_{21}T_1^{-1}$. Then $\rho(T_1) \cap \rho(T) = \rho(T_1) \cap \rho(\Psi T_1) \setminus \{0\}$ and for any λ in this set*
$$R(\lambda, T) = \begin{pmatrix} \lambda^{-1}(I + T_1R(\lambda, \Psi T_1)) & \lambda^{-1}T_1R(\lambda, \Psi T_1)T_1^{-1}T_{12} \\ \lambda^{-1}T_{21}R(\lambda, \Psi T_1) & \lambda^{-1}(I + T_{21}R(\lambda, \Psi T_1)T_1^{-1}T_{12}) \end{pmatrix}. \tag{2.1}$$

Proof. For any $\lambda \in \rho(T_1)$, let $S(\lambda) = \lambda - T_2 - T_{21}R(\lambda, T_1)T_{12}$ and let $\rho(S)$ denote the set of all $\lambda \in \mathbb{C}$ such that $S(\lambda)^{-1}$ is a bounded linear operator in X_2. By [9, Proposition H], we have $\rho(T_1) \cap \rho(T) = \rho(S)$ and for any λ in this set
$$R(\lambda, T) = \begin{pmatrix} R(\lambda, T_1)(I + T_{12}S^{-1}(\lambda)T_{21}R(\lambda, T_1)) & R(\lambda, T_1)T_{12}S^{-1}(\lambda) \\ S^{-1}(\lambda)T_{21}R(\lambda, T_1) & S^{-1}(\lambda) \end{pmatrix}. \tag{2.2}$$

Using the hypothesis $T_2 = T_{21}T_1^{-1}T_{12}$, we can rewrite $S(\lambda)$ in the form
$$S(\lambda) = \lambda - T_{21}T_1^{-1}T_{12} - T_{21}R(\lambda, T_1)T_{12} = \lambda\big(I - T_{21}R(\lambda, T_1)T_1^{-1}T_{12}\big).$$
We claim that $\rho(T_1) \cap \rho(T) = \rho(T_1) \cap \rho(\Psi T_1) \setminus \{0\}$ and for any λ in this set
$$S(\lambda)^{-1} = \lambda^{-1}\big(I + T_{21}R(\lambda, \Psi T_1)T_1^{-1}T_{12}\big). \tag{2.3}$$
Indeed, from the matrix operator identity
$$\begin{pmatrix} \lambda - T_1 & -T_{12} \\ -T_{21} & \lambda - T_2 \end{pmatrix} = \begin{pmatrix} I & -T_1^{-1}T_{12} \\ O & I \end{pmatrix} \begin{pmatrix} \lambda - T_1 - T_1^{-1}T_{12}T_{21} & O \\ -T_{21} & \lambda \end{pmatrix} \begin{pmatrix} I & T_1^{-1}T_{12} \\ O & I \end{pmatrix}$$

it follows that $\rho(T) = \rho(\Psi T_1) \setminus \{0\}$. For any $\lambda \in \rho(T_1) \cap \rho(\Psi T_1) \setminus \{0\}$, let $Z(\lambda) = \lambda^{-1}(I + T_{21} R(\lambda, \Psi T_1) T_1^{-1} T_{12})$. Now, we easily check that

$$R(\lambda, \Psi T_1) - R(\lambda, T_1) = R(\lambda, T_1) T_1^{-1} T_{12} T_{21} R(\lambda, \Psi T_1).$$

Hence

$$S(\lambda)Z(\lambda) = I + T_{21}\big(R(\lambda, \Psi T_1) - R(\lambda, T_1) - R(\lambda, T_1) T_1^{-1} T_{12} T_{21} R(\lambda, \Psi T_1)\big) T_1^{-1} T_{12} = I.$$

Analogously, using that $R(\lambda, \Psi T_1) - R(\lambda, T_1) = R(\lambda, \Psi T_1) T_1^{-1} T_{12} T_{21} R(\lambda, T_1)$, we show that $Z(\lambda)S(\lambda) = I$. Therefore, $Z(\lambda) = S(\lambda)^{-1}$. Finally, by substituting (2.3) in (2.2) we obtain (2.1). □

We now consider the group inverse of an operator with a matrix representation as in Theorem 2.1. The following result is a generalization of [1, Theorem 7.7.7] which was established for partitioned matrices.

Theorem 2.2. *Let $T \in \mathcal{B}(X)$ with the operator matrix form $T = \begin{pmatrix} T_1 & T_{12} \\ T_{21} & T_2 \end{pmatrix}$, where T_1 is invertible and $T_2 = T_{21} T_1^{-1} T_{12}$. Let $\Psi = I + T_1^{-1} T_{12} T_{21} T_1^{-1}$. Then T is group invertible if and only if Ψ is invertible. In this case,*

$$T^\sharp = \begin{pmatrix} (\Psi T_1 \Psi)^{-1} & (\Psi T_1 \Psi)^{-1} T_1^{-1} T_{12} \\ T_{21} T_1^{-1} (\Psi T_1 \Psi)^{-1} & T_{21} T_1^{-1} (\Psi T_1 \Psi)^{-1} T_1^{-1} T_{12} \end{pmatrix} \tag{2.4}$$

and

$$T^\pi = \begin{pmatrix} I - \Psi^{-1} & -\Psi^{-1} T_1^{-1} T_{12} \\ -T_{21} T_1^{-1} \Psi^{-1} & I - T_{21} T_1^{-1} \Psi^{-1} T_1^{-1} T_{12} \end{pmatrix}. \tag{2.5}$$

Proof. By Theorem 2.1 we have that, for any $\lambda \in \rho(T_1) \cap \rho(T) = \rho(T_1) \cap \rho(\Psi T_1) \setminus \{0\}$, the resolvent $R(\lambda, T)$ has the matrix operator representation (2.1). Suppose that T is group invertible, $\operatorname{ind}(T) = 1$. Using (1.2), we have

$$R(\lambda, T) = \frac{1}{\lambda}\big(I - TT^D\big) - \sum_{n=0}^{\infty} \lambda^n \big(T^D\big)^{n+1}, \tag{2.6}$$

in the region $0 < |\lambda| < (r(T^D))^{-1}$. On the other hand, since $0 \in \operatorname{iso} \sigma(T)$ and T_1 is invertible, there exists a punctured disk $D_{r_0} = \{\lambda \in \mathbb{C}\colon 0 < |\lambda| < r_0\}$ such that $D_{r_0} \subset \rho(T) \cap \rho(T_1) = \rho(T_1) \cap \rho(\Psi T_1) \setminus \{0\}$. Hence either $0 \in \rho(\Psi T_1)$ or $0 \in \operatorname{iso} \sigma(\Psi T_1)$. Therefore,

$$R(\lambda, \Psi T_1) = \sum_{n=-\infty}^{\infty} \lambda^n X_n,$$

for $\lambda \in D_{r_0}$ and X_n are defined as usual in Laurent series development. Using the expansion (2.6) in the left-hand side of (2.1) and the above expansion in the right-hand side of the same identity we deduce that $X_{-n} = 0$ for all $n \geqslant 1$ and, thus, ΨT_1 is invertible.

Conversely, suppose that ΨT_1 is invertible. Then we have the expansion

$$R(\lambda, \Psi T_1) = -\sum_{n=0}^{\infty} \lambda^n \big((\Psi T_1)^{-1}\big)^{n+1},$$

in the region $|\lambda| < r(\Psi T_1))$. Using this together with (2.1) we see that 0 is a simple pole of $R(\lambda, T)$ and taking coefficient of λ^0 and λ^{-1} we deduce the representation of $T^\sharp$ and T^π given in (2.4) and (2.5), respectively. □

3. Characterizations of group invertible perturbations

The following well-known result will be used throughout this paper.

Lemma 3.1. *Let $S, T \in \mathcal{B}(X)$. Then $I + ST$ is invertible in $\mathcal{B}(X)$ if and only if $I + TS$ is invertible in $\mathcal{B}(X)$.*

We begin with a characterization of an AG-inverse of the perturbation of a Drazin invertible operator for further use.

Theorem 3.2. *Let $A \in \mathcal{B}(X)$ be Drazin invertible with $\operatorname{ind}(A) = r$. The following assertions on $B \in \mathcal{B}(X)$ such that $I + A^D(B - A)$ is invertible are equivalent:*

(a) $Z = (I + A^D(B - A))^{-1}A^D = A^D(I + (B - A)A^D)^{-1}$ *is an AG-inverse of B.*
(b) $B(I + A^D(B - A))^{-1}A^\pi = A^\pi(I + (B - A)A^D)^{-1}B = 0$.
(c) $\mathcal{R}(B) \cap \mathcal{N}(A^r) = \{0\}$.
(d) $X = \mathcal{N}(B) + \mathcal{R}(A^r)$.
(e) $\mathcal{R}(BA^D) = \mathcal{R}(B)$.
(f) $\mathcal{N}(A^D B) = \mathcal{N}(B)$.
(g) $A^\pi(\mathcal{N}(B)) = \mathcal{N}(A^r)$.

Proof. (a) $\Leftrightarrow$ (b) Observe first that $A^\pi(A^\pi + A^D B)^{-1} = A^\pi$ and thus

$$ZBZ = \left(A^\pi + A^D B\right)^{-1}\left(A^D B + A^\pi - A^\pi\right)\left(A^\pi + A^D B\right)^{-1}A^D = Z.$$

Further,

$$BZB = B - B\left(I + A^D(B - A)\right)^{-1}A^\pi = B - A^\pi\left(I + A^D(B - A)\right)^{-1}B.$$

Hence $BZB = B$ if and only if $B(I + A^D(B - A))^{-1}A^\pi = A^\pi(I + A^D(B - A))^{-1}B = 0$. The equivalence between (a) and (b) is established.

Now, assume (a), or, equivalently, (b). Then BZ is a projection, $\mathcal{R}(BZ) = \mathcal{R}(B)$ and $\mathcal{N}(BZ) = \mathcal{N}(Z)$. Furthermore, we have $X = \mathcal{R}(BZ) \oplus \mathcal{N}(BZ) = \mathcal{R}(B) \oplus \mathcal{N}(Z)$. Clearly, $\mathcal{N}(Z) = \mathcal{N}(A^D) = \mathcal{N}(A^r)$. Consequently $\mathcal{R}(B) \cap \mathcal{N}(A^r) = \{0\}$. Using that $\mathcal{R}(BZ) = \mathcal{R}(BA^D)$ we also conclude that $\mathcal{R}(BA^D) = \mathcal{R}(B)$. The necessity of (c) and (e) is established. Similarly, we have that ZB is a projection, $\mathcal{R}(ZB) = \mathcal{R}(Z) = \mathcal{R}(A^D)$, $\mathcal{N}(A^D B) = \mathcal{N}(ZB) = \mathcal{N}(B)$ and $X = \mathcal{R}(ZB) \oplus \mathcal{N}(ZB) = \mathcal{R}(A^D) \oplus \mathcal{N}(B)$. The necessity of (d) and (f) is proved. Let $x \in \mathcal{N}(A^r) = \mathcal{N}(A^D)$. Write $x = A^D y + z$ relative to the direct sum $X = \mathcal{R}(A^D) \oplus \mathcal{N}(B)$. Then $x = A^\pi x = A^\pi z$ with $z \in \mathcal{N}(B)$. This proves (g).

Conversely, first assume (c). We have

$$A^D B\left(I + A^D(B - A)\right)^{-1} = \left(A^D B + A^\pi - A^\pi\right)\left(I + A^D(B - A)\right)^{-1} = I - A^\pi. \tag{3.1}$$

Therefore for arbitrary $x \in X$ we have $B(I + A^D(B - A))^{-1}A^\pi x \in \mathcal{R}(B) \cap \mathcal{N}(A^D)$. Hence

$$B\left(I + A^D(B - A)\right)^{-1}A^\pi x = 0 \quad \text{for each } x \in X.$$

This proves (b). Assume (d) and let $x \in X$ be written as $x = A^D y + z$ relative to the sum $X = \mathcal{R}(A^r) \oplus \mathcal{N}(B)$. Then

$$A^\pi\left(I + (B - A)A^D\right)^{-1}Bx = A^\pi\left(I + (B - A)A^D\right)^{-1}BA^D y = 0.$$

If (e) holds, then, for arbitrary $x \in X$, we have $Bx = A^D By$, for some $y \in X$. Therefore

$$A^\pi\left(I + (B - A)A^D\right)^{-1}Bx = A^\pi\left(I + (B - A)A^D\right)^{-1}A^D By = 0.$$

This proves (b). If (f) holds, in view of equality in (3.1), we conclude $B(I + A^D(B - A))^{-1}A^\pi = 0$.

Assume (g). For arbitrary $x \in X$ we have $A^\pi x = A^\pi u$ with $u \in \mathcal{N}(B)$. Then

$$B\left(I + A^D(B - A)\right)^{-1}A^\pi x = A^\pi\left(I + (B - A)A^D\right)^{-1}Bu = 0$$

and, thus, (b) holds. The proof is finished. □

Lemma 3.3. *(See [11, Theorem 5.2, (i) $\Leftrightarrow$ (ii)].) Let $F, G \in \mathcal{B}(X)$ be oblique projections. Then*

$$F - G \text{ is invertible} \quad \Leftrightarrow \quad X = \mathcal{R}(F) \oplus \mathcal{R}(G) \quad \text{and} \quad X = \mathcal{N}(F) \oplus \mathcal{N}(G).$$

The following theorem is the principal result in this section.

Theorem 3.4. *Let $A \in \mathcal{B}(X)$ be Drazin invertible with* $\operatorname{ind}(A) = r$. *The following assertions on $B \in \mathcal{B}(X)$ such that B is group invertible are equivalent*:

(i) $I + A^D(B - A)$ *is invertible and* $\mathcal{R}(B) \cap \mathcal{N}(A^r) = \{0\}$.
(ii) *Relative to the space decomposition $X = \mathcal{R}(A^r) \oplus \mathcal{N}(A^r)$, B has the matrix operator form*

$$B = \begin{pmatrix} B_1 & B_{12} \\ B_{21} & B_{21}B_1^{-1}B_{12} \end{pmatrix}, \quad B_1,\ I + B_1^{-1}B_{12}B_{21}B_1^{-1} \text{ invertible in } \mathcal{B}(\mathcal{R}(A^r)). \tag{3.2}$$

(iii) $I - A^\pi - B^\pi$ *is invertible.*
(iv) $X = \mathcal{N}(B) \oplus \mathcal{R}(A^r)$ *and* $X = \mathcal{R}(B) \oplus \mathcal{N}(A^r)$.
(v) $B^\pi + A^D B$ *is invertible and* $\mathcal{N}(B) \cap \mathcal{R}(A^r) = \{0\}$.
(vi) $\mathcal{R}(A^D) = \mathcal{R}(A^D BA^D)$, $\mathcal{N}(A^D B) = \mathcal{N}(B)$ *and* $\mathcal{N}(B) \cap \mathcal{R}(A^D) = \{0\}$.

Proof. (i) $\Rightarrow$ (ii). Write $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ and $B = \begin{pmatrix} B_1 & B_{12} \\ B_{21} & B_2 \end{pmatrix}$, with respect to the space decomposition $X = \mathcal{R}(A^r) \oplus \mathcal{N}(A^r)$. Then

$$I + A^D(B - A) = \begin{pmatrix} A_1^{-1}B_1 & A_1^{-1}B_{12} \\ 0 & I \end{pmatrix}.$$

Since $I + A^D(B - A)$ is invertible in $\mathcal{B}(X)$, it follows that B_1 is invertible in $\mathcal{B}(\mathcal{R}(A^r))$. Moreover, $(I + A^D(B - A))^{-1} = \begin{pmatrix} B_1^{-1}A_1 & -B_1^{-1}B_{12} \\ 0 & I \end{pmatrix}$. Using this representation, we obtain

$$B\big(I + A^D(B - A)\big)^{-1}A^\pi = \begin{pmatrix} B_1 & B_{12} \\ B_{21} & B_2 \end{pmatrix}\begin{pmatrix} 0 & -B_1^{-1}B_{12} \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & B_2 - B_{21}B_1^{-1}B_{12} \end{pmatrix}.$$

Now, note that under assumption (i) the relation $B(I + A^D(B - A))^{-1}A^\pi = 0$ also holds by Theorem 3.2, equivalence of (b) and (c). Consequently, $B_2 = B_{21}B_1^{-1}B_{12}$. Further, since B is group invertible, it follows from Theorem 2.2 that $I + B_1^{-1}B_{12}B_{21}B_1^{-1}$ is invertible in $\mathcal{B}(\mathcal{R}(A^r))$.

(ii) $\Rightarrow$ (iii). Let $\Psi := I + B_1^{-1}B_{12}B_{21}B_1^{-1}$. Applying Theorem 2.2, formula (2.5), we obtain

$$I - A^\pi - B^\pi = \begin{pmatrix} \Psi^{-1} & \Psi^{-1}B_1^{-1}B_{12} \\ B_{21}B_1^{-1}\Psi^{-1} & -I + B_{21}B_1^{-1}\Psi^{-1}B_1^{-1}B_{12} \end{pmatrix}.$$

Further,

$$\big(I - A^\pi - B^\pi\big)\big(I + A^D(B - A)\big)^{-1} = \begin{pmatrix} \Psi^{-1}B_1^{-1}A_1 & 0 \\ B_{21}B_1^{-1}\Psi^{-1}B_1^{-1}A_1 & -I \end{pmatrix}.$$

From this identity we conclude that $I - A^\pi - B^\pi$ is invertible.

(iii) $\Leftrightarrow$ (iv). Under assumption that B is group invertible, the spectral projection associated with $\{0\}$, B^π is defined and verifies $\mathcal{R}(B^\pi) = \mathcal{N}(B)$ and $\mathcal{N}(B^\pi) = \mathcal{R}(B)$. Since $I - A^\pi$ is the oblique projection with $\mathcal{R}(I - A^\pi) = \mathcal{R}(A^r)$ and $\mathcal{N}(I - A^\pi) = \mathcal{N}(A^r)$, this equivalence follows applying Lemma 3.3 with $F = I - A^\pi$ and $G = B^\pi$.

(iv) $\Rightarrow$ (v). First, we note that the operator $I + B^\pi A^D B$ is invertible in $\mathcal{B}(X)$. Now, by the identity

$$B^\pi + A^D B = \big(B^\pi + B^\sharp BA^D B\big)\big(I + B^\pi A^D B\big) \tag{3.3}$$

it follows that $B^\pi + A^D B$ is invertible if and only if $B^\pi + B^\sharp BA^D B$ is invertible. To prove that $B^\pi + B^\sharp BA^D B$ is one-to-one, suppose that $(B^\pi + B^\sharp BA^D B)x = 0$. Then $B^\pi x = -B^\sharp BA^D Bx$ and hence we derive that $B^\pi x = 0$ and $BA^D Bx = 0$. From the latter relation it follows that $A^D Bx \in \mathcal{N}(B) \cap \mathcal{R}(A^r)$ and, thus, $A^D Bx = 0$. But now $Bx \in \mathcal{R}(B) \cap \mathcal{N}(A^D)$ and so $Bx = 0$. This together with $B^\pi x = 0$ gives $x = B^\pi x + B^\sharp Bx = 0$.

To prove that $B^\pi + B^\sharp BA^D B$ is onto, let $x \in X$ be arbitrary. Since $X = \mathcal{N}(B) \oplus \mathcal{R}(A^r)$, $x = z + A^D y$ with $z \in \mathcal{N}(B)$ and $y \in X$. Thus $B^\sharp Bx = B^\sharp BA^D y$. Further, since $X = \mathcal{R}(B) \oplus \mathcal{N}(A^r)$, we can write $y = Bu + v$ with $u \in X$ and $v \in \mathcal{N}(A^r)$. Therefore, $B^\sharp Bx = B^\sharp BA^D y = B^\sharp BA^D Bu$ and

$$x = BB^\sharp x + B^\pi x = \big(B^\pi + B^\sharp BA^D B\big)\big(B^\sharp Bu + B^\pi x\big).$$

(v) $\Rightarrow$ (vi). Since $\mathcal{R}(A^D)=\mathcal{R}(A^r)$, the relation $\mathcal{N}(B)\cap\mathcal{R}(A^D)=\{0\}$ is clear. To prove the inclusion $\mathcal{N}(A^DB)\subseteq\mathcal{N}(B)$, suppose $x\in\mathcal{N}(A^DB)$. Then $(B^\pi+A^DB)BB^\sharp x=A^DBx=0$. Hence it follows that $BB^\sharp x=0$ and, thus, $x\in\mathcal{N}(B)$. The inclusion $\supseteq$ is obvious. It remains to prove that $\mathcal{R}(A^D)\subseteq\mathcal{R}(A^DBA^D)$ because the converse inclusion is clear. Let $x\in\mathcal{R}(A^D)$. Then there is $y\in X$ for which $x=(B^\pi+A^DB)y$. Hence $B^\pi y\in\mathcal{N}(B)\cap\mathcal{R}(A^D)$ and, thus $B^\pi y=0$. This implies that $x=A^DBy=A^DB(B^\pi+A^DB)z=A^DBA^DBz$ with $z=(B^\pi+A^DB)^{-1}y$, so that $x\in\mathcal{R}(A^DBA^D)$.

(vi) $\Rightarrow$ (i). In view of the identity

$$A^\pi+A^DB=\left(I+A^DBA^\pi\right)\left(A^\pi+A^DBA^DA\right)\tag{3.4}$$

it is enough to prove that $A^\pi+A^DBA^DA$ is invertible because $I+A^DBA^\pi$ is invertible in $\mathcal{B}(X)$. So we will prove that

$$\mathcal{N}\left(A^\pi+A^DBA^DA\right)=\{0\}\quad\text{and}\quad\mathcal{R}\left(A^\pi+A^DBA^DA\right)=X.$$

In view of (3.4), and using $\mathcal{N}(A^\pi)\cap\mathcal{N}(A^DB)=\{0\}$, we conclude $\mathcal{N}(A^\pi+A^DBA^DA)=\mathcal{N}(A^\pi+A^DB)=\{0\}$.

Towards the second identity let $x\in X$ be arbitrary. Since $\mathcal{R}(A^D)=\mathcal{R}(A^DBA^D)$, we can write $A^DAx=A^DBA^Dy$ for some $y\in X$ and, then

$$x=A^\pi x+A^DAx=\left(A^\pi+A^DBA^DA\right)\left(A^\pi x+A^Dy\right).$$

Further, by Theorem 3.2, equivalence of (f) and (c), it follows that $\mathcal{R}(B)\cap\mathcal{N}(A^r)=\{0\}$. $\square$

Next, we state an auxiliary result, which will be used in the next section to give a condition for the continuity of the group inverse.

The following characterization of group invertible elements is due to C. Schmoeger [17, Theorem 3.3].

Lemma 3.5. *$T\in\mathcal{B}(X)$ is group invertible $\Leftrightarrow$ for some inner inverse S of T the operator $I-TS-ST$ is invertible in $\mathcal{B}(X)$.*

Lemma 3.6. *Let $A\in\mathcal{B}(X)$ be Drazin invertible, with* $\operatorname{ind}(A)=r$, *and $B\in\mathcal{B}(X)$ such that $I+A^D(B-A)$ is invertible and $\mathcal{R}(B)\cap\mathcal{N}(A^r)=\{0\}$. Then*

$$B\text{ is group invertible}\quad\Leftrightarrow\quad I+A^D\left(B^2-A^2\right)A^D\text{ is invertible in }\mathcal{B}(X).$$

Proof. By Theorem 3.2 we have that $Z=(I+A^D(B-A))^{-1}A^D=A^D(I+(B-A)A^D)^{-1}$ is an AG-inverse of B. Since

$$\left(A^\pi-A^DB\right)(I-BZ-ZB)\left(A^\pi+BA^D\right)=A^\pi+A^DB^2A^D,$$

it follows that $I-A^D(B^2-A^2)A^D$ is invertible if and only if $I-BZ-ZB$ is invertible. Now assume that B is group invertible. Then $X=\mathcal{R}(B)\oplus\mathcal{N}(B)=\mathcal{N}(I-BZ)\oplus\mathcal{N}(ZB)$. On the other hand, we have $X=\mathcal{R}(A^D)\oplus\mathcal{N}(A^D)=\mathcal{R}(I-BZ)\oplus\mathcal{R}(ZB)$. By Lemma 3.3 we conclude that $I-BZ-ZB$ is invertible and, thus, $I+A^D(B^2-A^2)A^D$ is invertible. The converse part follows from Lemma 3.5. $\square$

4. Perturbation bounds for Drazin inverses and spectral projectors

First we establish an error bound for Drazin inverses of the class of perturbations that we study in terms of the norm of the difference of the spectral projections associated with $\{0\}$. We observe that condition $\|A^D(B-A)\|<1$ implies that $I+A^D(B-A)$ is invertible.

Lemma 4.1. *Let $A,B\in\mathcal{B}(X)$ be Drazin invertible. If $\|B^\pi-A^\pi\|<1$, then $\mathcal{N}(B^\pi)\cap\mathcal{N}(I-A^\pi)=\{0\}$.*

Proof. Since $\|A^\pi-B^\pi\|<1$, we derive that $I-A^\pi+B^\pi$ is invertible. Now, assume that $x\in\mathcal{N}(B^\pi)\cap\mathcal{N}(I-A^\pi)$. Then $(I-A^\pi)x=B^\pi x=0$, and thus $(I-A^\pi+B^\pi)x=0$. Hence $x=0$ and the implication is proved. $\square$

Theorem 4.2. *Let $A, B \in \mathcal{B}(X)$ be Drazin invertible and group invertible, respectively, and let $E = B - A$. If $\|A^D E\| + \|B^\pi - A^\pi\| < 1$, then*

$$\frac{\|B^\sharp - A^D\|}{\|A^D\|} \leqslant \frac{\|A^D E\| + 2\|B^\pi - A^\pi\|}{1 - \|A^D E\| - \|B^\pi - A^\pi\|}. \tag{4.1}$$

Proof. By Lemma 4.1, we have $\mathcal{N}(B^\pi) \cap \mathcal{N}(I - A^\pi) = \{0\}$ or, equivalently, $\mathcal{R}(B) \cap \mathcal{N}(A^r) = \{0\}$ with $r = \operatorname{ind}(A)$. By the hypothesis we also have that $I + A^D E$ is invertible. Therefore, we can apply Theorem 3.4 and from the equivalence of (i) and (v) it follows that $B^\pi + A^D B$ is invertible in $\mathcal{B}(X)$. Now, note that the following identity holds:

$$B^\sharp = \left(B^\pi + A^D B\right)^{-1} A^D \left(I - B^\pi\right) = \left(I + A^D E + B^\pi - A^\pi\right)^{-1} A^D \left(I - B^\pi + A^\pi\right).$$

This yields that

$$B^\sharp - A^D = -\left(A^D E + B^\pi - A^\pi\right)\left(B^\sharp - A^D + A^D\right) - A^D\left(B^\pi - A^\pi\right),$$

and, hence,

$$\left\|B^\sharp - A^D\right\| \leqslant \left(\left\|A^D E\right\| + \left\|B^\pi - A^\pi\right\|\right)\left\|B^\sharp - A^D\right\| + \left(\left\|A^D E\right\| + 2\left\|B^\pi - A^\pi\right\|\right)\left\|A^D\right\|.$$

Using assumption $\|A^D E\| + \|B^\pi - A^\pi\| < 1$, we derive the upper bound (4.1) from this estimate. □

The bound (4.1) can be combined with an explicit upper bound for $\|B^\pi - A^\pi\|$. Indeed, if Δ is an upper bound of $\|B^\pi - A^\pi\|$ such that $\|A^D E\| + \Delta < 1$, then

$$\frac{\|B^\sharp - A^D\|}{\|A^D\|} \leqslant \frac{\|A^D E\| + 2\Delta}{1 - \|A^D E\| - \Delta}, \tag{4.2}$$

which follows by repeating the argument for the proof of the above theorem.

An estimate for the norm of the difference of two idempotents operators in terms of the gap between their null spaces and ranges was obtained in [3, Proposition 2.2].

The rest of this section is devoted to formulations of explicit error bounds for the spectral projection associated with $\{0\}$ and the Drazin inverse. Our analysis will be based on the matrix form of the perturbed operator provided by Theorem 3.4.

We start by giving a technical result which will be used in the sequel.

Lemma 4.3. *Let $A \in \mathcal{B}(X)$ be Drazin invertible. Assume that $B \in \mathcal{B}(X)$ has a matrix operator form as in* (3.2), *and let $E = B - A$, then*

$$\begin{aligned}
\mathcal{U}_E &:= \left(I + A^D E\right)^{-1} A^D E A^\pi = \begin{pmatrix} 0 & B_1^{-1} B_{12} \\ 0 & 0 \end{pmatrix}, \\
\mathcal{L}_E &:= A^\pi E A^D \left(I + E A^D\right)^{-1} = \begin{pmatrix} 0 & 0 \\ B_{21} B_1^{-1} & 0 \end{pmatrix}, \\
(I + \mathcal{U}_E \mathcal{L}_E)^{-1} &= \begin{pmatrix} (I + B_1^{-1} B_{12} B_{21} B_1^{-1})^{-1} & 0 \\ 0 & I \end{pmatrix}.
\end{aligned} \tag{4.3}$$

If $\max\{\|A^D E\|, \|E A^D\|\} < \frac{1}{1+\sqrt{\|A^\pi\|}}$, *then*

$$\|\mathcal{U}_E\| \leqslant \frac{\|A^D E A^\pi\|}{1 - \|A^D E\|}, \qquad \|\mathcal{L}_E\| \leqslant \frac{\|A^\pi E A^D\|}{1 - \|E A^D\|}, \tag{4.4}$$

$$\left\|(I + \mathcal{U}_E \mathcal{L}_E)^{-1}\right\| \leqslant \frac{(1 - \|A^D E\|)(1 - \|E A^D\|)}{(1 - \|A^D E\|)(1 - \|E A^D\|) - \|A^D E A^\pi\| \|A^\pi E A^D\|}. \tag{4.5}$$

Proof. The matrix form of the operator $E = B - A$ with respect to the decomposition $X = \mathcal{R}(A^r) \oplus \mathcal{N}(A^r)$ is given by

$$E = x\begin{pmatrix} B_1 - A_1 & B_{12} \\ B_{21} & B_{21}B_1^{-1}B_{12} - A_2 \end{pmatrix}, \quad B_1,\ I + B_1^{-1}B_{12}B_{21}B_1^{-1} \text{ invertible in } \mathcal{R}(A^r). \tag{4.6}$$

Then

$$A^D E A^\pi = \begin{pmatrix} 0 & A_1^{-1}B_{12} \\ 0 & 0 \end{pmatrix}, \qquad A^\pi E A^D = \begin{pmatrix} 0 & 0 \\ B_{21}A_1^{-1} & 0 \end{pmatrix}.$$

On the other hand,

$$\left(I + A^D E\right)^{-1} = \begin{pmatrix} B_1^{-1}A_1 & -B_1^{-1}B_{12} \\ 0 & I \end{pmatrix}, \qquad \left(I + EA^D\right)^{-1} = \begin{pmatrix} A_1 B_1^{-1} & 0 \\ -B_{21}B_1^{-1} & I \end{pmatrix}. \tag{4.7}$$

Using the above representations we easily see that (4.3) hold.

Next we assume that $\max\{\|EA^D\|, \|A^D E\|\} < \frac{1}{1+\sqrt{\|A^\pi\|}}$ $(\leqslant \frac{1}{2})$. It follows from the two elementary estimates

$$\left\|\left(I + A^D E\right)^{-1}\right\| \leqslant \frac{1}{1 - \|A^D E\|}, \qquad \left\|\left(I + EA^D\right)^{-1}\right\| \leqslant \frac{1}{1 - \|EA^D\|}$$

that the upper bounds of $\|\mathcal{U}_E\|$ and $\|\mathcal{L}_E\|$ in (4.4) hold.

Using the estimates (4.4), we derive that

$$\|\mathcal{U}_E\mathcal{L}_E\| \leqslant \frac{\|A^D E A^\pi\|\|A^\pi E A^D\|}{(1 - \|A^D E\|)(1 - \|EA^D\|)} \leqslant \frac{\|A^D E\|\|EA^D\|\|A^\pi\|^2}{(1 - \|A^D E\|)(1 - \|EA^D\|)} < 1,$$

and, combining the inequality $\|(I + \mathcal{U}_E\mathcal{L}_E)^{-1}\| \leqslant \frac{1}{1-\|\mathcal{U}_E\mathcal{L}_E\|}$ with the above upper bound of $\|\mathcal{U}_E\mathcal{L}_E\|$ we obtain the estimate (4.4). □

Theorem 4.4. *Let $A, B \in \mathcal{B}(X)$ be Drazin invertible and group invertible, respectively, and let $E = B - A$. If $\mathcal{R}(B) \cap \mathcal{N}(A^r) = \{0\}$ with $r = \operatorname{ind}(A)$ and $\max\{\|A^D E\|, \|EA^D\|\} < \frac{1}{1+\sqrt{\|A^\pi\|}}$, then*

$$B^\pi = \left(I + A^D E\right)^{-1}\left(A^\pi - \mathcal{L}_E(I + \mathcal{U}_E\mathcal{L}_E)^{-1}(I + \mathcal{U}_E)\right), \tag{4.8}$$

where $\mathcal{U}_E$ and $\mathcal{L}_E$ are defined as in (4.3) *and*

$$\left\|B^\pi - A^\pi\right\| \leqslant \frac{\|A^D E A^\pi\|}{1 - \|A^D E\|} + \frac{\|A^\pi E A^D\|}{\nu(E)}\left(1 + \frac{\|A^D E A^\pi\|}{1 - \|A^D E\|}\right), \tag{4.9}$$

where $\nu(E) = (1 - \|A^D E\|)(1 - \|EA^D\|) - \|A^D E A^\pi\|\|A^\pi E A^D\|$.

Proof. According to Theorem 3.4, (i) ⇔ (ii), B has a matrix operator form as in (3.2). Further, by denoting $\Psi = I + B_1^{-1}B_{12}B_{21}B_1^{-1}$, Theorem 2.2, formula (2.5), yields

$$A^\pi B^\pi = \begin{pmatrix} 0 & 0 \\ -B_{21}B_1^{-1}\Psi^{-1} & I - B_{21}B_1^{-1}\Psi^{-1}B_1^{-1}B_{12} \end{pmatrix}.$$

On the other hand, using (4.3) we see that

$$A^\pi - \mathcal{L}_E(I + \mathcal{U}_E\mathcal{L}_E)^{-1}(I + \mathcal{U}_E) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ B_{21}B_1^{-1} & 0 \end{pmatrix}\begin{pmatrix} \Psi^{-1} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & B_1^{-1}B_{12} \\ 0 & I \end{pmatrix} = A^\pi B^\pi.$$

This, together with the fact that $(I + A^D E)B^\pi = A^\pi B^\pi$ because $BB^\pi = 0$, proves (4.8). Now, we can write

$$B^\pi - A^\pi = -A^D E\left(B^\pi - A^\pi + A^\pi\right)\mathcal{L}_E(I + \mathcal{U}_E\mathcal{L}_E)^{-1}(I + \mathcal{U}_E).$$

Hence, taking norms and using the estimates (4.4) we derive

$$\left\|B^\pi - A^\pi\right\| \leqslant \frac{\|A^D E A^\pi\|}{1 - \|A^D E\|} + \frac{\|A^\pi E A^D\|\|(I + \mathcal{U}_E\mathcal{L}_E)^{-1}\|}{(1 - \|A^D E\|)(1 - \|EA^D\|)}\left(1 + \frac{\|A^D E A^\pi\|}{1 - \|A^D E\|}\right).$$

Finally replacing $\|(I + \mathcal{U}_E\mathcal{L}_E)^{-1}\|$ by its upper bound defined in (4.5) we obtain (4.9). □

An upper bound of $\|B^\sharp - A^D\|/\|A^D\|$ can be obtained by applying (4.2) with the right-hand side of (4.9) in place of Δ. Next, we give a direct estimation for the relative error of the Drazin inverse.

Theorem 4.5. *Let $A, B \in \mathcal{B}(X)$ be Drazin invertible and group invertible, respectively, and let $E = B - A$. If $\mathcal{R}(B) \cap \mathcal{N}(A^r) = \{0\}$ with $\operatorname{ind}(A) = r$ and $\max\{\|A^D E\|, \|EA^D\|\} < \frac{1}{1+\sqrt{\|A^\pi\|}}$, then*

$$B^\sharp = \left(I + A^D E\right)^{-1}\left(A^D + A^D \Sigma_2 + \Sigma_1 A^D + \Sigma_1 A^D \Sigma_2\right), \tag{4.10}$$

$$\Sigma_1 := \mathcal{L}_E (I + \mathcal{U}_E \mathcal{L}_E)^{-1}\left(I - \left(I + A^D E\right)^{-1} A^D E\right), \qquad \Sigma_2 := (I + \mathcal{U}_E \mathcal{L}_E)^{-1} \mathcal{U}_E (I - \mathcal{L}_E), \tag{4.11}$$

and

$$\frac{\|B^\sharp - A^D\|}{\|A^D\|} \leqslant \frac{\|A^D E\|}{1 - \|A^D E\|} + \delta_1 + \delta_2 + \delta_1 \delta_2, \tag{4.12}$$

$$\delta_1 := \frac{\|A^\pi E A^D\|}{\nu(E)}\left(1 + \frac{\|A^D E\|}{1 - \|A^D E\|}\right), \qquad \delta_2 := \frac{\|A^D E A^\pi\|(1 - \|EA^D\|)}{\nu(E)(1 - \|A^D E\|)}\left(1 + \frac{\|A^\pi E A^D\|}{1 - \|EA^D\|}\right), \tag{4.13}$$

where $\nu(E) := (1 - \|A^D E\|)(1 - \|EA^D\|) - \|A^D E A^\pi\| \|A^\pi E A^D\|$.

Proof. According to Theorem 3.4, (i)⇔(ii), B has a matrix operator form as in (3.2). Further, by denoting $\Psi = I + B_1^{-1} B_{12} B_{21} B_1^{-1}$, Theorem 2.2, formula (2.4), yields

$$B^\sharp = \begin{pmatrix} (\Psi B_1 \Psi)^{-1} & (\Psi B_1 \Psi)^{-1} B_1^{-1} B_{12} \\ B_{21} B_1^{-1} (\Psi B_1 \Psi)^{-1} & B_{21} B_1^{-1} (\Psi B_1 \Psi)^{-1} B_1^{-1} B_{12} \end{pmatrix}. \tag{4.14}$$

Hence, with $E = B - A$ written in the form (4.6), we obtain that

$$\left(I + A^D E\right) B^\sharp = \begin{pmatrix} A_1^{-1} \Psi^{-1} & A_1^{-1} \Psi^{-1} B_1^{-1} B_{12} \\ B_{21} B_1^{-1} (\Psi B_1 \Psi)^{-1} & B_{21} B_1^{-1} (\Psi B_1 \Psi))^{-1} B_1^{-1} B_{12} \end{pmatrix}. \tag{4.15}$$

Using representations (4.3) and (4.7) provided by Lemma 4.3, we can write the operators Σ_1 and Σ_2, which have been defined in (4.11), in the form

$$\Sigma_1 = \begin{pmatrix} 0 & 0 \\ B_{21} B_1^{-1} & 0 \end{pmatrix} \begin{pmatrix} \Psi^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} B_1^{-1} A_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ B_{21} B_1^{-1} \Psi^{-1} B_1^{-1} A_1 & 0 \end{pmatrix},$$

$$\Sigma_2 = \begin{pmatrix} \Psi^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & B_1^{-1} B_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -B_{21} B_1^{-1} & I \end{pmatrix} = \begin{pmatrix} -I + \Psi^{-1} & \Psi^{-1} B_1^{-1} B_{12} \\ 0 & 0 \end{pmatrix}.$$

We derive (4.10) seeing that the right-hand side of (4.15) is equal to the matrix operator which results after computing $A^D + A^D \Sigma_2 + \Sigma_1 A^D + \Sigma_1 A^D \Sigma_2$ using the above representations.

Now, Eq. (4.10) implies

$$B^\sharp - A^D = -A^D E\left(B^\sharp - A^D + A^D\right) + A^D \Sigma_2 + \Sigma_1 A^D + \Sigma_1 A^D \Sigma_2.$$

Hence,

$$\left\|B^\sharp - A^D\right\| \leqslant \left\|A^D E\right\| \left\|B^\sharp - A^D\right\| + \left\|A^D\right\| \left(\left\|A^D E\right\| + \|\Sigma_1\| + \|\Sigma_2\| + \|\Sigma_1\| \|\Sigma_2\|\right),$$

and, since $\|A^D E\| < 1$, it follows that

$$\frac{\|B^\sharp - A^D\|}{\|A^D\|} \leqslant \frac{\|A^D E\|}{1 - \|A^D E\|} + \frac{\|\Sigma_1\| + \|\Sigma_2\| + \|\Sigma_1\| \|\Sigma_2\|}{1 - \|A^D E\|}. \tag{4.16}$$

Now, using the bounds in (4.4) and (4.5), the terms Σ_1 and Σ_2 are estimated as follows:

$$\|\Sigma_1\| \leqslant \|\mathcal{L}_E\| \left\|(I + \mathcal{U}_E \mathcal{L}_E)^{-1}\right\| \left(1 + \left\|\left(I + A^D E\right)^{-1} A^D E\right\|\right) \leqslant \frac{\|A^\pi E A^D\|(1 - \|A^D E\|)}{\nu(E)}\left(1 + \frac{\|A^D E\|}{1 - \|A^D E\|}\right)$$

and

$$\|\Sigma_2\| \leqslant \left\|(I + \mathcal{U}_E \mathcal{L}_E)^{-1}\right\| \|\mathcal{U}_E\| \left(1 + \|\mathcal{L}_E\|\right) \leqslant \frac{\|A^D E A^\pi\|(1 - \|A^D E\|)}{\nu(E)}\left(1 + \frac{\|A^\pi E A^D\|}{1 - \|EA^D\|}\right).$$

Finally, the upper bound (4.12), with δ_1 and δ_2 defined as in (4.13), is inferred by substituting these estimates in (4.16). □

We remark that hypothesis $\mathcal{R}(B) \cap \mathcal{N}(A^r) = \{0\}$ in the previous perturbation results can be replaced by any of the equivalent conditions given in Theorem 3.2.

From [15] we know that for $A_n, A \in \mathcal{B}(X)$ Drazin invertible operators such that $A_n \to A$, there is equivalence

$$A_n^D \to A^D \quad \Leftrightarrow \quad A_n^\pi \to A^\pi.$$

Next, we give a necessary and sufficient condition for the continuity of the group inverse of bounded operators in Banach spaces.

Theorem 4.6. *Let $A \in \mathcal{B}(X)$ be group invertible and let $A_n \in \mathcal{B}(X)$ operators such that $A_n \to A$. Then the following assertions are equivalent*:

(a) *A_n has a group inverse for n large enough and $A_n^\sharp \to A^\sharp$.*
(b) *$\mathcal{R}(A_n) \cap \mathcal{N}(A) = \{0\}$ for n large enough.*

Proof. Since $A_n \to A$, for n large enough we have $\|A^\sharp\|\|A_n - A\| < 1$ and $\|A^\sharp\|^2\|A_n^2 - A^2\| < 1$, thus, the operators $I + A^\sharp(A_n - A)$ and $I + A^\sharp(A_n^2 - A^2)A^\sharp$ are invertible.

Assume (a). Since $A_n^\sharp \to A^\sharp$, then $A_n^\pi \to A^\pi$. Hence $\|A_n^\pi - A^\pi\| < 1$ for n large enough. Now, by Lemma 4.1, $\mathcal{N}(A_n^\pi) \cap \mathcal{N}(I - A^\pi) = \{0\}$ or, equivalently, $\mathcal{R}(A_n) \cap \mathcal{N}(A) = \{0\}$.

Assume (b). It follows from Lemma 3.6 that A_n is group invertible for n large enough. Also we have $\|A^\sharp\|\|A_n - A\| < \frac{1}{1+\sqrt{\|A^\pi\|}}$ for n large enough and the estimate (4.12)–(4.13) given in Theorem 4.5 leads to $\|A_n^\sharp - A^\sharp\| \to 0$. $\square$

References

S.L. Campbell, C.D. Meyer, Generalized Inverses of Linear Transformations, Dover, New York, 1991; Originally published: Pitman, London, 1979.

S.R. Caradus, Operator Theory of the Generalized Inverse, Queen's Papers in Pure and Appl. Math., vol. 38, Queen's University, Kingston, ON, 1974.

N. Castro-González, J.J. Koliha, V. Rakočević, Continuity and general perturbation of the Drazin inverse for closed linear operators, Abstr. Appl. Anal. 6 (2002) 335–347.

N. Castro-González, J.J. Koliha, Y. Wei, Error bounds for perturbation of the Drazin inverse of closed operators with equal spectral projections, Appl. Anal. 81 (2002) 915–928.

N. Castro-González, J. Robles, J.Y. Veléz-Cerrada, Characterizations of a class of matrices and perturbation of the Drazin inverse, preprint.

N. Castro-González, J.Y. Vélez-Cerrada, Elements in rings and Banach algebras with related spectral idempotents, J. Aust. Math. Soc. 80 (2006) 383–396.

D.S. Djordjević, W. Wei, Operators with equal projections related to their generalized inverses, Appl. Math. Comput. 155 (2004) 655–664.

B. Duggal, R. Harte, In Ho Jeon, Polaroid operators and Weyl's theorem, Proc. Amer. Math. Soc. 132 (5) (2003) 1345–1349.

G. Hengstberger, Spectral properties of operator matrices, Tübinger Berichte zur Funktionalanalysis 1 (1991/92) 65–72.

J.J. Koliha, V. Rakočević, Continuity of the Drazin inverse II, Studia Math. 131 (1998) 167–177.

J.J. Koliha, V. Rakočević, Invertibility of the difference of idempotents, Linear Multilinear Algebra 388 (2004) 279–288.

C.D. Meyer, The role of the group generalized inverse in the theory of finite Markov chains, SIAM Rev. 17 (1975) 443–464.

C.D. Meyer, The condition of a finite Markov chain and perturbation bounds for the limiting probabilities, SIAM J. Algebraic Discrete Methods 1 (1980) 273–283.

M.Z. Nashed, Inner, outer, and generalized inverses in Banach and Hilbert spaces, Numer. Funct. Anal. Optim. 9 (384) (1987) 261–325.

V. Rakočević, Continuity of the Drazin inverse, J. Operator Theory 41 (1999) 55–68.

V. Rakočević, Y. Wei, The perturbation theory for the Drazin inverse and its aplications II, J. Aust. Math. Soc. 70 (2001) 189–197.

C. Schmoeger, On a class of generalized Fredholm operators I, Demonstratio Math. XXX (4) (1997) 829–842.

C. Schmoeger, On the operator equations $ABA = A^2$ and $BAB = B^2$, Publ. Inst. Math. 78 (92) (2005) 127–133.

D.J. Spitzner, T.R. Boucher, Asymptotic variance of functionals of discrete time Markov chains via the Drazin inverse, Electron. Comm. Probab. 12 (2007) 120–133.

Y. Wei, S. Qiao, The representation and ap oximation of the Drazin inverse of a linear operator in Hilbert space, Appl. Math. Comput. 138 (2003) 77–89.